Int. J. Heat Mass Transfer. Vol. 14, pp. 1575–1588. Pergamon Press 1971. Printed in Great Britain

EFFECT OF LONGITUDINAL SURFACE CURVATURE ON HEAT TRANSFER WITH DISSIPATION

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(*Received* 22 *April* 1970 *and in revised form* 30 *December* 1970)

Abstract—Heat transfer including dissipation on a surface with longitudinal curvature has been analyzed for forced convection in laminar flow, by the method of matched asymptotic expansions. Using the classical Falkner–Skan wedge flows as the first order solution to the momentum equation, the first order solution to the energy equation including dissipation has been obtained. Then, by extending the analysis, the second order perturbation for the velocity and temperature fields is obtained. The analysis permits the wall temperature to vary as a power function of the distance when there is no dissipation; however, when dissipation is included, the wall temperature variation is determined by the pressure gradient parameter if similar solutions are still required. The ordinary differential equations obtained from the similarity analysis have been numerically solved. The calculated second order temperature profiles have been presented graphically as functions of the pressure gradient parameter, Prandtl number, Eckert number, wall temperature distribution parameter and surface curvature. It is seen that the second order effect is considerable for conditions close to separation and are not necessarily negligible compared with first order effects. Dissipation can considerably affect heat transfer for fluids with high Prandtl numbers, the Nusselt number changing from positive to negative, as the Eckert number changes from zero to unity. Further, at any given Prandtl number, as the curvature changes from concave to convex, the Nusselt number decreases if the Eckert number is small, while it increases if the Eckert number is close to unity.

NOMENCLATURE

A, a constant for prescribed wall temperature;

C, a constant for prescribed velocity distribution at the edge of the boundary layer;

C_1, a constant to prescribe local surface curvature;

E, Eckert number;

f, non-dimensional stream function;

P, non-dimensional pressure in the outer flow;

p, non-dimensional pressure in the inner flow;

Pr, Prandtl number $\mu C_p/\bar{K}$;

C_p, specific heat at constant pressure;

$K(x)$, local surface curvature;

$\bar{K}$, thermal conductivity;

k, curvature parameter;

Re, Reynolds number $(U_\infty \rho L/\mu)$;

L, characteristic length;

Nu, Nusselt number, $hL/\bar{K}$;

h, film coefficient of heat transfer;

T, non-dimensional temperature in the outer flow;

T_w, specified wall surface temperature;

t, non-dimensional temperature in the inner flow;

U_{os}, inviscid surface velocity;

U, non-dimensional velocity in the outer flow;

u, non-dimensional tangential velocity in the inner flow;

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V, non-dimensional normal velocity in the outer flow;
v, non-dimensional normal velocity in the inner flow;
X, outer independent variable;
x, inner independent variable;
Y, outer independent variable;
y, inner independent variable;
S, wall surface on which flow takes place.

Greek symbols
β, pressure gradient parameter;
Φ, dissipation function;
ρ, fluid density;
μ, absolute viscosity;
γ, wall temperature distribution parameter;
θ, non-dimensional temperature;
ε, $(Re)^{-\frac{1}{2}}$;
ψ, stream function;
α, limit of $(\eta - f_1)$ as $\eta \to \infty$;
η, proportional to stream function at the edge of the boundary layer.

Subscripts
x, local quantity;
1, first order quantity;
2, second order quantity;
w, value at wall;
∞, free stream quantity.

1. INTRODUCTION

THE CALCULATION of heat transfer from curved solid surfaces is of interest in a large number of problems of practical interest. Some examples of such problems are the aerodynamic heating of bodies in flight, flow along airfoil surfaces as in turbines and compressors, the cooling of gas turbine blades, flow through rocket nozzles, etc.

In the study of flow over surfaces with small curvature, it is well known that the boundary layer equations of momentum and energy can be solved to determine both skin friction and heat transfer, yielding results independent of curvature. In other words, to the limit of accuracy of boundary layer theory, skin friction and heat transfer are unaffected by curvature. The solutions are therefore the same as for flow over the Falkner–Skan wedge. The only effect of curvature is to determine the inviscid surface speed to which the viscous velocity should tend as the edge of the boundary layer is approached. If the solution is continued up to the second order, curvature appears as a parameter in the differential equations for momentum, and the solution therefore depends explicitly on curvature, unlike in the classical solution, where the dependence was merely implicit.

The influence of curvature on skin friction has been studied by Murphy [1], Hayasi [2], Yen and Toba [3] and Narasimha and Ojha [4]. It has been observed that longitudinal curvature reduces skin friction, if the flow is on the convex side of the surface. Van Dyke [5] has observed a similar decrease in skin friction in the study of flow over a parabola. The problem of heat transfer from wedges (without dissipation) has been considered by Levy [6] using first order boundary layer arguments. Schultz–Grunow and Breuer [7] have considered the problem of constant wall temperature at zero pressure gradient on a curved surface and have concluded that convex longitudinal curvature reduces heat transfer.

In the present paper, the problem of heat transfer including dissipation from a surface with longitudinal curvature has been considered. As in Levy's [6] analysis for wedge flow, both the inviscid surface speed of the fluid and the wall temperature are allowed to vary as power functions of distance from the start. The surface curvature is limited to small values by the requirements $K(x) \ll 1/\delta$ and $K(x) \ll 1/\delta_t$. Following the singular perturbation scheme in a manner similar to that of Narasimha and Ojha [4], solutions have been obtained to the heat transfer problem up to the second order, including dissipation. All the equations have been numerically integrated for a wide range of parameters, $0{\cdot}7 \leqslant Pr \leqslant 100$, and $-0{\cdot}195 \leqslant \beta \leqslant 2$.

For the heat transfer problem without dissipation, the wall temperature parameter γ is arbitrary, and has been allowed to take on values in the range $-1{\cdot}0 \leqslant \gamma \leqslant 4{\cdot}0$. For the problem including dissipation, similarity requires that γ should depend on β, and cannot be chosen arbitrarily. The Eckert number appears as a new parameter and numerical solutions have been obtained for values of Eckert number ranging from 0 to 1. All the solutions have been graphically presented.

2. ANALYSIS

The continuity, momentum and energy equations for the steady flow of an incompressible constant property fluid may be written in the following non-dimensional form:

$$\operatorname{div} \boldsymbol{U} = 0 \tag{1}$$

$$\boldsymbol{U}.\operatorname{grad} \boldsymbol{U} = -\operatorname{grad} P - \frac{1}{Re}\operatorname{curl}\operatorname{curl} \boldsymbol{U} \tag{2}$$

$$\boldsymbol{U}.\operatorname{grad} T = \frac{1}{RePr}\nabla^2 T + \frac{1}{Re}\frac{U_\infty^2}{C_p T_\infty}\Phi. \tag{3}$$

In these equations, $\boldsymbol{U}$ is the vector velocity at any point in the flow field, P is the thermodynamic pressure and T is the absolute temperature while Re and Pr are the Reynolds and Prandtl numbers respectively. The quantity Φ is the dissipation function given by the expression

$$\Phi = \tfrac{1}{2}[e_{11}^2 + e_{22}^2 + 2e_{12}^2].$$

e_{11}, e_{22} and e_{12} being the rate of strain components in the X–Y plane. As usual, the free stream speed U_∞, free stream temperature T_∞, a characteristic dimension L, as well as the dynamic pressure $\frac{1}{2}\rho U_\infty^2$ have been used to obtain the non-dimensional form presented in equations (1)–(3). Though written for fluids with constant densities, these equations may themselves be used even for the study of all flow situations where the Mach number is small and compressibility effects negligible. The boundary conditions are

$$\text{Far upstream: } \boldsymbol{U} \to 1; \quad T \to 1 \tag{4a}$$

$$\text{Surface } S: \quad \boldsymbol{U} = 0; \quad T = T_w(x)/T_\infty. \tag{4b}$$

One first writes the outer expansions of the variables in equations (1)–(3) by considering the limit $\varepsilon = (Re)^{-\frac{1}{2}} \to 0$, holding X fixed. These expansions may be written as

$$\boldsymbol{U} = \boldsymbol{U}_1(X, Y) + \varepsilon \boldsymbol{U}_2(X, Y) + \ldots. \tag{5a}$$

$$P = P_1(X, Y) + \varepsilon P_2(X, Y) + \ldots. \tag{5b}$$

and

$$T = T_1(X, Y) + \varepsilon T_2(X, Y) + \ldots. \tag{5c}$$

It is readily seen from the above, after substituting in the differential equations and collecting the coefficients of various powers of ε that the outer flow is potential at least up to the second order. The solution to the outer equations may be made to satisfy the conditions given by equation (4a). The conditions valid near the wall have to be determined by writing an inner solution and then matching the outer and inner solutions to the required order.

In order to determine the inner solutions, one uses an orthogonal coordinate system consisting of curves parallel to the wall and lines perpendicular to the wall, with the origin at the front stagnation point. The inner coordinates are then written as $y = Y/\varepsilon$ and $x = X$. The inner expansions for the velocities, pressure and temperature may now be written in the form:

$$u = u_1(x, y) + \varepsilon u_2(x, y) + \ldots \tag{6a}$$

$$v = \varepsilon v_1(x, y) + \varepsilon^2 v_2(x, y) + \ldots \tag{6b}$$

$$p = p_1(x, y) + \varepsilon p_2(x, y) + \ldots \tag{6c}$$

$$t = t_1(x, y) + \varepsilon t_2(x, y) + \ldots \tag{6d}$$

The continuity, Navier–Stokes and energy equations then give the first order inner equations

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{7a}$$

$$\frac{\partial p_1}{\partial y} = 0 \tag{7b}$$

$$u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} = \frac{\partial^2 u_1}{\partial y^2} - \frac{\partial p_1}{\partial x} \tag{7c}$$

and

$$u_1 \frac{\partial t_1}{\partial x} + v_1 \frac{\partial t_1}{\partial y} = \frac{1}{Pr}\frac{\partial^2 t_1}{\partial y^2} + \frac{U_\infty^2}{C_p T_\infty}\left(\frac{\partial u_1}{\partial y}\right)^2. \tag{7d}$$

Equations of set (7) are the first order boundary layer equations of Prandtl along with the corresponding first order energy equation. As commented upon earlier, these equations do not contain curvature explicitly, so that to this order, the skin-friction and heat transfer are independent of the curvature parameter. It may be noticed further, on examining the energy equation (7d), that the order of the dissipation term depends on the magnitude of the quantity ($U_\infty^2/C_p T_\infty$). For gases, this quantity is equal to $(C_p - C_v)M^2/C_v$, where M is the Mach number and C_p, C_v are the specific heats at constant pressure and volume respectively. Hence, if the Mach number is small as in slow speed flow, dissipation is usually negligible in gases since they have Prandtl numbers on the order of unity. This argument does not apply to liquids whose Prandtl numbers range anywhere between 1 and 1000 or more. In such cases, one must take dissipation into consideration, even if the temperatures are moderate and the fluid incompressible.

Collecting terms of the second order (coefficients of ε) leads to the equations presented below:

$$\frac{\partial u_2}{\partial x} + \frac{\partial}{\partial y}(v_2 + Kyv_1) = 0 \tag{8a}$$

$$\frac{\partial p_2}{\partial y} - Ku_1^2 = 0 \tag{8b}$$

$$u_1 \frac{\partial u_2}{\partial x} + v_1 \frac{\partial u_2}{\partial y} + u_2 \frac{\partial u_1}{\partial x} + v_2 \frac{\partial u_1}{\partial y} + \frac{\partial p_2}{\partial x} - \frac{\partial^2 u_2}{\partial y^2} = K\left\{y\left(u_1 \frac{\partial u_1}{\partial x} + \frac{\partial p_1}{\partial x}\right) + \frac{\partial u_1}{\partial y} - u_1 v_1\right\} \tag{8c}$$

and,

$$u_1 \frac{\partial t_2}{\partial x} + v_1 \frac{\partial t_2}{\partial y} + u_2 \frac{\partial t_1}{\partial x} + v_2 \frac{\partial t_1}{\partial y} - \frac{1}{Pr}\frac{\partial^2 t_2}{\partial y^2} = K\left\{yu_1 \frac{\partial t_1}{\partial x} + \frac{1}{Pr}\frac{\partial t_1}{\partial y}\right\} + \frac{2U_\infty^2}{C_p T_\infty}\left\{\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial y} - Ku_1 \frac{\partial u_1}{\partial y}\right\}. \tag{8d}$$

The boundary conditions for the first and second order equations may be obtained by using the restricted matching principle of Lagerstrom [8]. The matching and initial conditions for the velocities, pressure and temperature are then obtained in the form:

First order:

$$y = 0:\ u_1(x, 0) = v_1(x, 0) = 0;\quad t_1(x, 0) = T_w/T_\infty \tag{9a}$$

$$y \to \infty:\ u_1(x, y) = U_1(X, 0);\quad V_1(X, 0) = 0 \tag{9b}$$

$$p_1(x) = P_1(X, 0) \tag{9c}$$

$$t_1(x, y) = T_1(X, 0). \tag{9d}$$

Second order:

$$y = 0:\ u_2(x, 0) = v_2(x, 0) = t_2(x, 0) = 0 \tag{10a}$$

$$y \to \infty:\ u_2(x, y) = U_2(X, 0) - KyU_1(X, 0) \tag{10b}$$

$$v_2(x, y) = V_2(X, 0) - \left|\frac{\partial U_1}{\partial X}\right|_{y=0} y \tag{10c}$$

$$p_2(x, y) = P_1(X, 0) + KyU_1^2(X, 0) \tag{10d}$$

$$t_2(x, y) = T_2(X, 0) = 0. \tag{10e}$$

It is observed from the equations of sets (7)–(10) that the first order momentum equation is the only non-linear equation in the system. The second order momentum and the energy equations (first as well as second) are linear and nonhomogeneous, so that the principle of superposition may be used to solve them. This means that if we determine a solution to the system containing only terms arising from curvature effects, and then another solution to the system containing only those terms arising from displacement effects and then superpose the two solutions, the composite solution will be complete in all aspects. Since the purpose of the present paper is to confine itself to curvature effects, all terms like $U_2(x, 0)$ and $V_2(X, 0)$ which arise due to displacement effects are removed from the following analysis.

In order to simplify calculations further, the two second order momentum equations (8b) and (8c) are combined after eliminating p_2 to yield [4],

$$u_1 \frac{\partial u_2}{\partial x} + v_1 \frac{\partial u_2}{\partial y} + u_2 \frac{\partial u_1}{\partial x} + v_2 \frac{\partial u_1}{\partial y} = -\frac{\partial}{\partial x}\{K[\int_0^y u_1^2 \mathrm{d}y + \int_0^\infty (U_{os}^2 - u_1^2)\mathrm{d}y]\} + \frac{\partial^2 u_2}{\partial y^2} + K\left\{y\left(u_1 \frac{\partial u_1}{\partial x} + \frac{\partial p_1}{\partial x}\right) + \frac{\partial u_1}{\partial y} - u_1 v_1\right\}, \quad (11)$$

where U_{os} represents the inviscid surface speed in wedge flow, obtained from potential theory. The boundary conditions for the second order equations are

$$y = 0: \quad u_2 = v_2 = 0; \quad t_2(x, 0) = 0 \quad (12a)$$

$$y \to \infty: \quad u_2(x, y) \to -KyU_{os}; \quad t_2(x, y) = T_2(X, 0) = 0. \quad (12b)$$

The equations in the present form are amenable to a similarity analysis, when the surface speed $U_{os}(x)$ and the wall temperature $T_w(x)$ are specified in a particular way. The similarity transformations are the same as those used by Narasimha and Ojha [4], extended to include the energy equation. Consider the transformations

$$\xi = \int_0^x U_{os}(x)\mathrm{d}x, \eta = (2\xi)^{-\frac{1}{2}} U_{os} y, \quad (13a)$$

$$\psi_1(x, y) = (2\xi)^{\frac{1}{2}} f_1(\eta) \text{ and } \theta_1(\eta) = \frac{t_1 - 1}{(T_w/T_\infty) - 1} \quad (13b)$$

where f_1 and θ_1 are functions of η alone. If these variables are substituted into the equations of set (7), it is seen that similar solutions are possible only if

$$m = \frac{\mathrm{d}(\ln U_{os})}{\mathrm{d}(\ln x)} = \text{const}; \quad U_{os} = Cx^m \quad (14a)$$

and

$$\gamma = \frac{\mathrm{d}[\ln(T_w - T_\infty)]}{\mathrm{d}(\ln x)} = \text{const}; \quad T_w(x) = T_\infty + Ax^\gamma, \quad (14b)$$

where C is an arbitrary constant and m, γ are parameters specifying the variation of inviscid surface speed and wall temperature respectively. If the dissipation effect introduced by the term $(U_\infty^2/C_p T_\infty)(\partial u_1/\partial y)^2$ is negligible as for fluids with sufficiently small Prandtl numbers, one obtains respectively for $f_1(\eta)$ and $\theta_1(\eta)$, the classical Falkner–Skan and related energy equation of Levy [6]:

$$f_1'''(\eta) + f_1(\eta)f_1''(\eta) + \beta[1 - f_1'^2(\eta)] = 0 \quad (15a)$$

$$\theta_1''(\eta) + Pr[f_1(\eta)\theta_1'(\eta) - \gamma(2 - \beta)f_1'(\eta)\theta_1(\eta)] = 0 \quad (15b)$$

where the primes denote successive differentiations with respect to η, while $\beta = 2m/(m + 1)$. In obtaining equations (15), both the quantities A and γ may be treated as arbitrary constants.

If dissipation cannot be neglected, restrictions have to be placed on the values of A and γ in order to reduce the energy equation to a form similar to equation (15b). It is easily verified, using the same transformations as above that if

$$\gamma = 2m = \frac{2\beta}{2 - \beta} \quad (16)$$

the energy equation including dissipation reduces to the form

$$\theta_1''(\eta) + Pr[f_1(\eta)\theta_1'(\eta) - 2\beta f_1'(\eta)\theta_1(\eta) + Ef_1''^2(\eta)] = 0 \quad (17)$$

where $E = C^2U_\infty^2/C_pA$ is the Eckert number. The Eckert number is zero in non-dissipative flow, and its magnitude in dissipative flow depends on both the free-stream speed U_∞, and the magnitudes of constants C and A. The Eckert number can be quite large when A is small.

Equations of set (15) and (17) have the boundary conditions

$$\eta = 0: \quad f_1(0) = f_1'(0) = 0; \; \theta_1(0) = 1 \quad (18a)$$

$$\eta \to \infty; \; f_1'(\eta) = 1; \; \theta_1(\eta) = 0. \quad (18b)$$

For the second order system, the use of the continuity equation allows the definition of the second order stream function given by the equations

$$u_2 = \frac{\partial\psi_2}{\partial y}; \quad v_2 = -\frac{\partial\psi_2}{\partial x} - Kyv_1. \quad (19)$$

If the curvature $K(x)$ is prescribed as a power function of x by the relation $K(x) = C_1x^l$ where C_1 is a constant and l is a parameter, it is possible to obtain similar solutions to the second order equations by defining the functions $f_2(\eta)$ and $\theta_2(\eta)$ according to the equations:

$$\psi_2 = (2\xi)^n f_2(\eta) \text{ and } \theta_2(\eta) = \frac{t_2}{(T_w/T_\infty) - 1}. \quad (20a)$$

The exponent n is then given by the relation $n = (l+1)/(m+1)$. In the following, attention will nonetheless be concentrated on a subset of the above general transformations, wherein it is required that the power of ξ in the equation for ψ_2 be the same as in equation (13b) for ψ_1. Such solutions of the first and second orders have been called "jointly similar" in the literature [9]. Then, $n = 1/2$ and $l = (m-1)/2$. On writing with Narasimha and Ojha [4]

$$K(x) = kU_{os}/(2\xi)^{\frac{1}{2}} = k\left[\frac{C(m+1)}{2}\right]^{\frac{1}{2}} x^{(m-1)/2} \quad (20b)$$

where k is called the "Curvature Parameter", one obtains the following second order equations:

$$f_2''' + f_1f_2'' - 2\beta f_1'f_2' + f_1''f_2 = k\{f_1''(\eta f_1 - 1) - f_1f_1' - \beta[\eta(f_1'^2 - 1) - \frac{2}{\beta+1}(f_1'' + f_1f_1' + \beta\eta + \alpha)]\} \quad (21a)$$

$$\theta_2'' + Pr[f_1\theta_2' - (2-\beta)\gamma(f_1'\theta_2 + f_2'\theta_1) + f_2\theta_1'] = k\{Pr[\eta f_1\theta_1' - \gamma\eta(2-\beta)f_1'\theta_1] - \theta_1'\}. \quad (21b)$$

Equation (21b) is the second order energy equation without dissipation. If dissipation is included, one gets the equation

$$\theta_2'' + Pr[f_1\theta_2' - 2\beta(f_1'\theta_2 + f_2'\theta_1) + f_2\theta_1'] = k\{Pr[\eta(f_1\theta_1' - 2\beta f_1'\theta_1) + 2Ef_1'f_1''] - \theta_1']\} - 2PrEf_1''f_2''. \quad (22)$$

In equation (21a) above, $\alpha = \lim_{\eta\to\infty}(\eta - f_1)$, and is a function of β. The boundary conditions may be written in the form

$$\eta = 0: \quad f_2(0) = f_2'(0); \quad \theta_2(0) = 0 \quad (23a)$$

$$\eta \to \infty: \; f_2'(\eta) = -k\eta; \; \theta_2(\infty) = 0. \quad (23b)$$

The set of equations (16), (17), (21) and (22) has been integrated numerically on a CDC 6400 digital computer, using the Runge–Kutta–Gill integration procedure. Solutions have been obtained for a large range of parameters, $0{\cdot}7 \leqslant Pr \leqslant 100$, $-0{\cdot}195 \leqslant \beta \leqslant 2{\cdot}0$, $0 \leqslant E \leqslant 1$, and $-1{\cdot}0 \leqslant \gamma \leqslant 4{\cdot}0$ (without dissipation). The computed results enable one to determine the local heat transfer coefficient, as well as the change in Nusselt number due to the second order correction. These are given respectively by the expressions

$$(Nu_x)(Re_x)^{-\frac{1}{2}} = -\frac{1}{(2-\beta)^{\frac{1}{2}}}\left[\theta_1'(0) + \varepsilon k\frac{\theta_2'(0)}{k}\right] \quad (24a)$$

$$(\Delta Nu_x)(Re_x)^{-\frac{1}{2}} = -\frac{k\varepsilon}{(2-\beta)^{\frac{1}{2}}}\frac{\theta_2'(0)}{k} \quad (24b)$$

3. DISCUSSION AND CONCLUSIONS

Equations (15), (17), (21) and (22) along with boundary conditions describe two dimensional flow past a curved surface and contain the parameters β, γ, Pr, E and k. In order to bring out the behaviour of the solutions clearly, flows without and with dissipation are considered separately in what follows.

(a) *Non-dissipative flows* ($E = 0$). The effect of longitudinal curvature on heat transfer in flow with no dissipation has been discussed in detail by Gupta [10]. A brief summary of the results is given below so that flows without and with dissipation may be compared.

Figure 1 shows the variation of wall temperature gradient $\theta_2'(0)/k$ as a function of β, for specified values of γ. The quantity $\theta_2'(0)/k$ is positive [while $\theta_1'(0)$ is negative], increasing monotonically as the flow changes from high acceleration to separation, for positive and small negative values of γ. The large increase in $\theta_2'(0)/k$ with the approach of separation is similar to the trend of $f_2''(0)/k$ [4], which becomes very large near separation, $\beta = -0{\cdot}198838$. Since the second order quantities become far larger than the corresponding first order quantities near separation, it is doubtful whether the asymptotic expansions are valid close to separation. Similarly, it is doubtful whether the solutions are valid for large negative values of γ, because of the singularity at $x = 0$ in the prescribed surface temperature $[T_w(x) = T_\infty + Ax^\gamma]$. For small negative values of γ and all positive values, the solutions are expected to be uniformly valid.

Figure 2 shows the variation in local Nusselt number with the curvature parameter, and compares the present results with those of Schultz-Grunow and Breuer [7] and Van Dyke [5]. As is clear from equation (24a), the local Nusselt number is linearly related to $k\varepsilon$ when Pr, γ and β are fixed. With adverse pressure gradients (β negative), the Nusselt number diminishes more rapidly than for favourable pressure gradients (β positive), as $k\varepsilon$ increases. For highly accelerated flows, the Nusselt number varies very little with changing $k\varepsilon$, i.e. $\theta_2'(0)/k$ is small for such cases. For plane stagnation flow, ($\gamma = 0$, $\beta = 1$ and $Pr = 0{\cdot}7$), the results of Fig. 2 are in exact agreement with

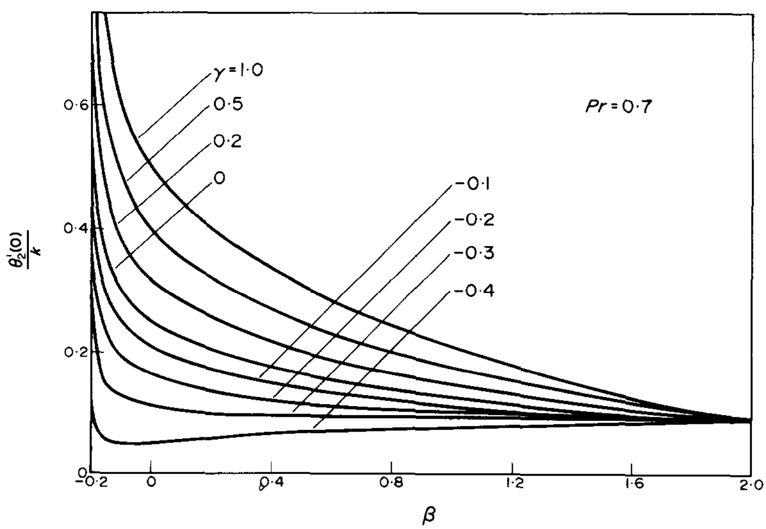

FIG. 1. Variation in second order wall temperature gradient ($E = 0$) with pressure gradient.

Van Dyke [5], while for flow with zero pressure gradient ($\beta = 0$), constant wall temperature ($\gamma = 0$) and $Pr = 0{\cdot}7$, the agreement between the present results and those of Schultz-Grunow and Breuer [7] is not exact. In fact, though the plot of [7] appears to be straight in Fig. 2, there is a slight curvature in it, making it deviate more and more from the presently calculated results as the magnitude of εk increases. It has been shown by Gustafson and Pelech [11] that

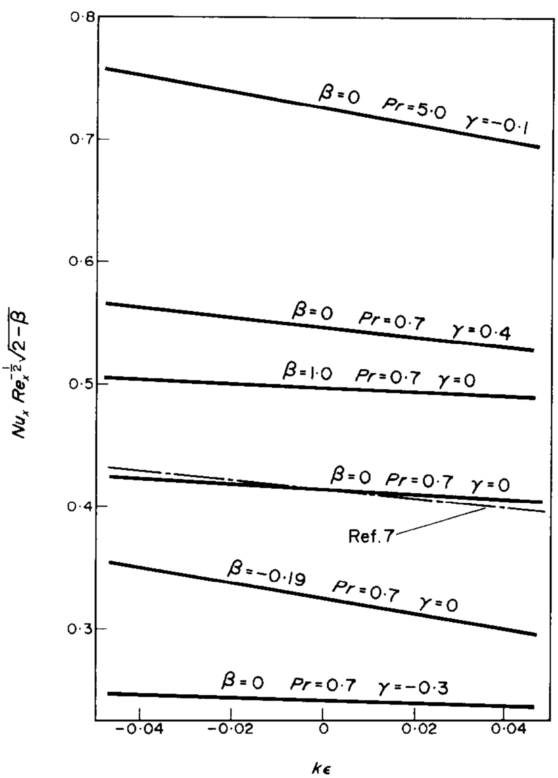

FIG. 2. Variation in local Nusselt number ($E = 0$) with curvature parameter.

the results of [7] are approximate due to the use of the inviscid surface speed rather than the true velocity in the boundary layer to calculate the pressure gradient in the normal direction.

If the change in Nusselt number due to second order effects is computed, it is seen to be rather small (1·5 per cent or less) in highly accelerated flows. In flows with adverse pressure gradients however, the second order effects can range from 0 to 14 per cent or more, as shown by Gupta [10]. Such large differences cannot be neglected, and the second order effects must be taken into account for accurate predictions of heat transfer.

(b) *Flows with dissipation.* In general, dissipation may be expected to reduce heat transfer from the wall to the fluid for all Prandtl numbers. The effects of pressure gradient parameter on the first order temperature profile are indicated in Fig. 3, which is a plot of $\theta_1(\eta)$ with β as a parameter for two values of Prandtl number, $Pr = 0{\cdot}7$, and $Pr = 10{\cdot}0$ and Eckert number unity. Because of the large Eckert number chosen, they exaggerate the effects of dissipation. Nevertheless, they clearly show that dissipation changes the wall temperature gradient, and may cause reversed heat transfer from the fluid to the wall, even when the wall temperature is higher than that of the free-stream. Many of the temperature profiles, especially those for conditions close to separation, exhibit a point of inflection.

The effect of dissipation on the second order temperature profile is exhibited in Fig. 4, which shows $\theta_2(\eta)/k$ for various values of β, Eckert number unity and $Pr = 0{\cdot}7$. It is interesting that these curves exhibit a negative gradient at the wall (i.e. $\theta_2'(0)/k$ is negative), as opposed to the case with no dissipation, where the gradient is positive. For Eckert numbers intermediate between zero and unity, $\theta_2'(0)/k$ changes gradually from positive to negative values. Even when $\theta_2(\eta)/k$ starts with negative values, it becomes positive, before reducing to zero for sufficiently large η at the edge of the boundary layer. As in non-dissipative flow, the second order effects are large for negative values of β close to separation and small for highly accelerated flows. A similar trend is observed for higher values of Prandtl number as well, except that the magnitudes of $\theta_2(\eta)/k$ are much larger.

Figure 5 shows the first order temperature gradients $\theta_1'(0)$ as a function of β, with Eckert

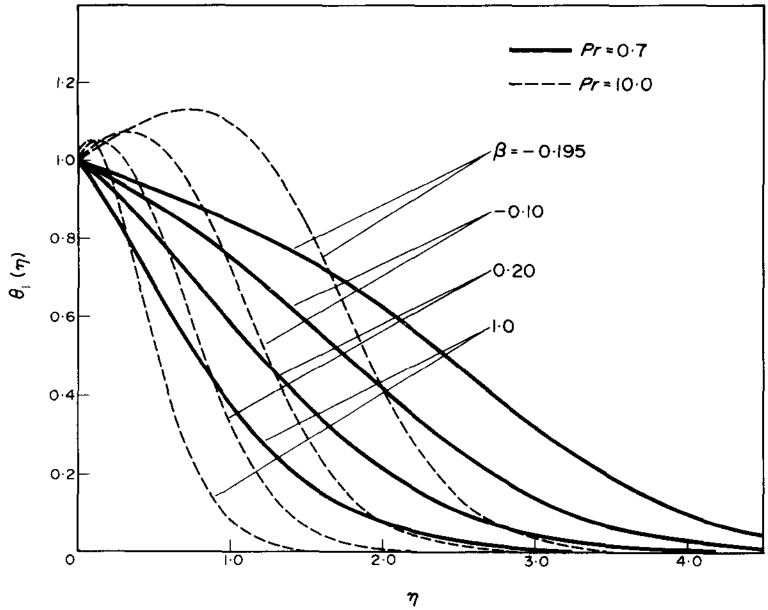

FIG. 3. First order temperature profiles including dissipation, with $E = 1$.

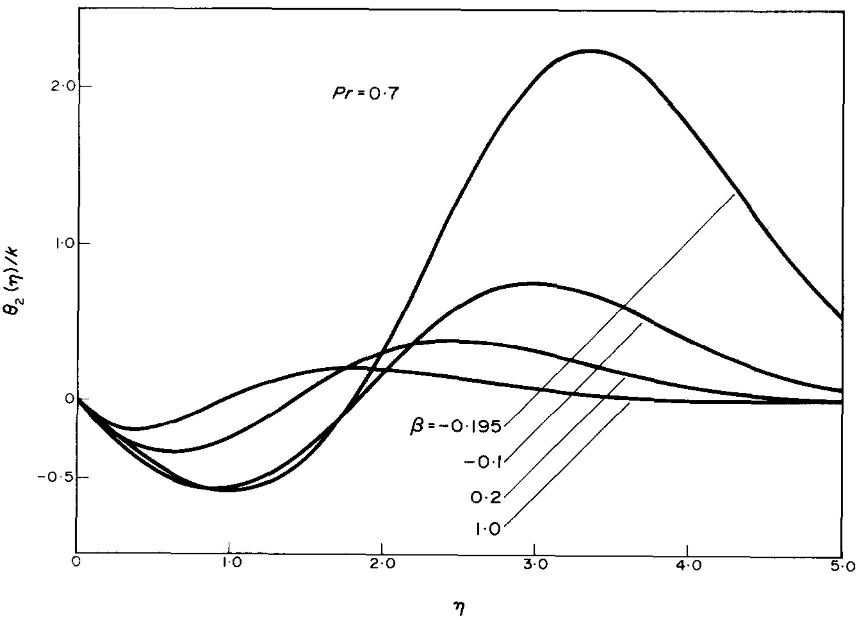

FIG. 4. Second order temperature profiles including dissipation, with $E = 1$.

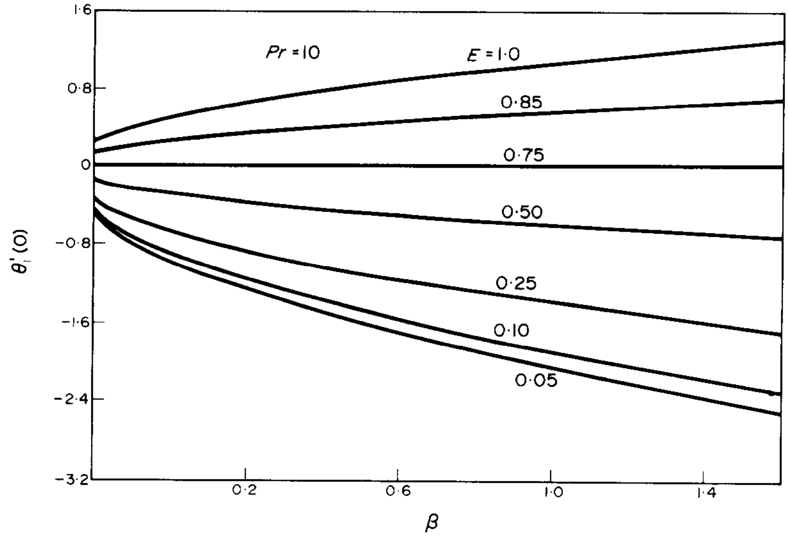

FIG. 5. Variation of $\theta_1'(0)$ with pressure gradient parameter, for various values of E.

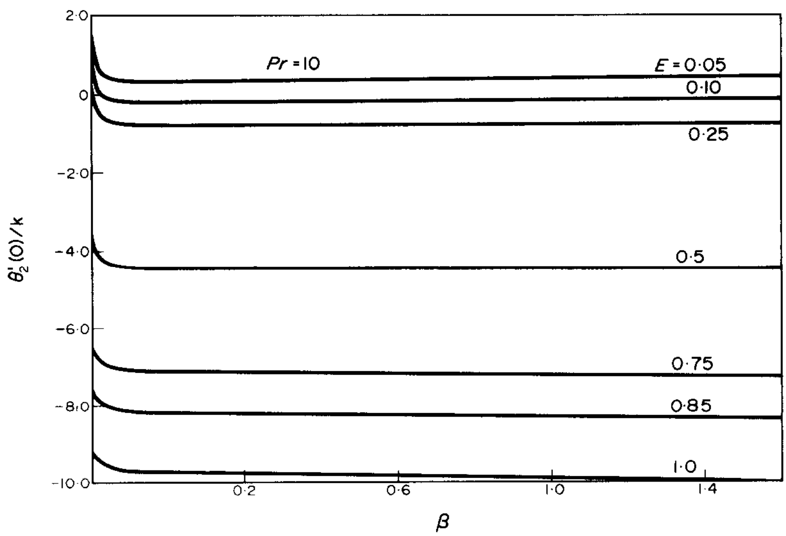

FIG. 6. Variation of $\theta_2'(0)/k$ with pressure gradient parameter, for various values of E.

number as a parameter, all for a Prandtl number of 10. The temperature gradient which is seen to be negative when the Eckert number is small, gradually decreases in magnitude and becomes almost zero when $E = 0{\cdot}75$. For larger values of Eckert number, the gradient becomes positive, showing that heat transfer occurs from the fluid to the wall, even though the plate temperature is higher than that of the fluid free-stream. Figure 6 shows the variation of the second order gradient $\theta'_2(0)/k$ as a function of β, again with Eckert number as a parameter, for a Prandtl number of 10. When Eckert number increases, $\theta'_2(0)/k$ changes sign from positive to negative, as opposed to the first order gradient which changes from negative to positive. This difference may be explained by referring to equations (7d) and (8d). In equation (7d), the term $(\partial u_1/\partial y)^2$ is positive, and the first order dissipation term raises the temperature of the fluid close to the wall above that of the fluid with no dissipation. This in turn reduces heat transfer from the wall to the fluid, making $\theta'_1(0)$ decrease in magnitude first, and finally change sign from negative to positive for sufficiently large values of Eckert number. On the other hand, to the second order, dissipation effects are represented by the terms $(\partial u_1/\partial y)x\,(\partial u_2/\partial y)$—$Ku_1(\partial u_1/\partial y)$, as seen from equation (8d). These terms are positive if the curvature parameter k is negative and negative if k is positive, as may be deduced from the second order momentum equation. Hence, for negative values of k, the second order dissipation terms may be expected to reinforce the first order terms. On a body with concave curvature ($k < 0$) therefore, the first and second order terms should both reduce heat transfer from the warm surface to the fluid. In other words, the temperature gradients at the wall should be positive, or if negative, small in magnitude compared with the case of zero Eckert number. Similarly, if the body has convex curvature ($k > 0$), the first and second order terms are expected to cause opposing effects.

Figure 7 shows the gradients $\theta'_1(0)$ and

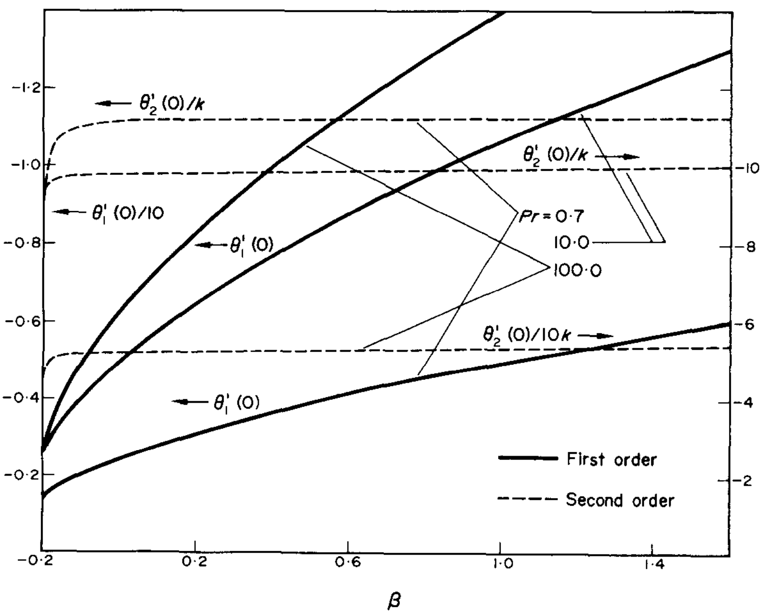

FIG. 7. Values of $\theta'_1(0)$ and $\theta'_2(0)/k$ with Prandtl number as parameter.

$\theta'_2(0)/k$ as functions of β for three values of Prandtl numbers, $Pr = 0{\cdot}7$, 10 and 100, so that the effect of Prandtl numbers on these gradients may be studied. All curves have been plotted with Eckert number 1·0. It is clear that $\theta'_1(0)$ changes sign from negative to positive with increasing values of Prandtl number, while $\theta'_2(0)/k$ is negative for all Prandtl numbers when Eckert number is unity. For small Eckert numbers, $\theta'_2(0)/k$ may be expected to be negative when $Pr = 0{\cdot}7$, and to become positive for large Prandtl numbers.

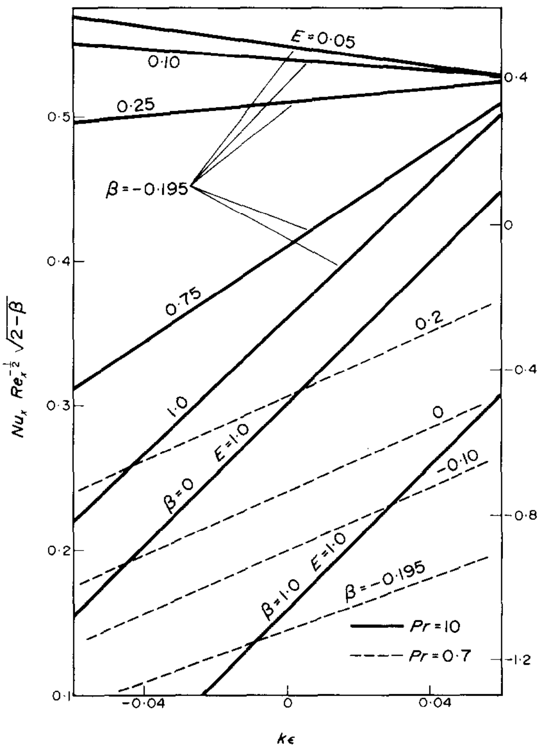

FIG. 8. Plot of $Nu_x(Re_x)^{-\frac{1}{2}}\sqrt{(2-\beta)}$ against the curvature parameter.

These effects are more clearly exhibited in Fig. 8, which shows plots of $(Nu_x)(Re_x)^{-\frac{1}{2}}\sqrt{(2-\beta)}$ against $k\varepsilon$, for selected values of β, Pr and E. The solid lines near the top of the figure are for Eckert numbers ranging from 0·05 to 1·0, all for $\beta = -0{\cdot}195$ and $Pr = 10$. It is quite clear that the slopes of the lines change from negative to positive as the Eckert number increases. This means that for small dissipation, heat transfer diminishes with change of curvature from concave to convex, while for large dissipation, the heat transfer increases with a similar change of curvature. Moreover, as observed earlier, for sufficiently large Eckert numbers, the heat transfer is from the fluid into the plate, when the curvature changes sign, and for a sufficiently large convex curvature, heat transfer again occurs from the hot surface to the fluid.

The dotted straight lines in Fig. 8 are for $Pr = 0{\cdot}7$ and varying values of β, all for Eckert number unity. All these lines have positive slopes, showing that even with relatively small Prandtl numbers, dissipation can cause an increase in heat transfer with a change of curvature from negative to positive.

The slope of the lines drawn in Fig. 8 indicates the change of Nusselt number due to curvature effect, as opposed to the first order effect which does not take account of curvature. The magnitudes of Nu_x along the line $k\varepsilon = 0$ are those due to the first order theory alone. It is readily seen that when Eckert number is small and β is close to separation, the relative change in Nusselt number due to curvature is about 12–15 per cent. For large Eckert numbers above 0·5, this relative change may be as high as 150 per cent or greater, when $k\varepsilon$ changes from 0·0 to 0·06. Nevertheless, there lies a small range of values of Eckert numbers between these two extremes, where the curvature parameter has negligible influence on heat transfer so that the first order theory is by itself sufficient. In the range of parameters indicated on the graph, this condition may be expected for Eckert numbers around 0·15, if the Prandtl number is 10. Further, the second order effects can be quite considerable at large Prandtl numbers, even when the pressure gradient is positive. This can be seen from the plots in Fig. 8 for other values of β. Thus, for large Prandtl numbers, the second order effects cannot be neglected, if reasonable approximations to heat transfer coefficient are necessary.

The results discussed above cannot be extrapolated to large values of curvature parameter since the theory that has so far been developed is valid only for small curvatures. For large values of εk, the expansions for velocity and temperature will not be uniformly valid.

It is possible, with the curves plotted earlier, to find the magnitude of the curvature parameter which reduces the wall to an adiabatic surface, correct to the second order. This is obtained by equating Nu_x in equation (24a) to zero when there is obtained

$$(k\varepsilon)_{\text{adiabatic wall}} = -\theta'_1(0)/[\theta'_2(0)/k]. \quad (25)$$

An examination of the curves in Figs. 6 and 7 shows for a Prandtl number of 10, that the value of $k\varepsilon$ which reduces the wall to adiabatic conditions is positive for very small Eckert numbers and becomes negative as the Eckert number rises, but then again becomes positive when the Eckert number approaches unity. This trend is generally true of higher Prandtl numbers as well. For a Prandtl number 0·7 on the other hand, the value of $k\varepsilon$ for an adiabatic surface is first positive when Eckert number is small, and is negative at values of Eckert number near unity. Moreover, the magnitude of curvature parameter which renders the surface adiabatic increases with increasing values of the pressure gradient parameter β. The values of $k\varepsilon$ needed for an adiabatic surface have not been calculated since they are large, and the asymptotic expansions are likely to be invalid for large curvatures.

Concluding, it is seen that for fluids of large Prandtl numbers, dissipation has a considerable effect, even when the Eckert number is not very large. The heat transfer characteristics of the wall vary considerably depending on the Eckert number, as is easily expected. When the flow is about to separate, the second order effects can be extremely pronounced, while they are not so severe for accelerating flows.

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EFFET D'UNE COURBURE LONGITUDINALE D'UNE SURFACE SUR UN TRANSFERT THERMIQUE AVEC DISSIPATION

Résumé—On a analysé par la méthod des développements asymptotiques un transfert thermique par convection forcé dans un écoulement laminaire avec dissipation sur une surface à courbure longitudinale. En prenant les écoulements classiques de Falkner-Skan autour d'un dièdre comme solution du premier ordre de l'équation de quantité de mouvement, on obtient alors la solution de premier ordre de l'équation de l'énergie avec dissipation. Puis par extension de l'analyse est obtenue la perturbation de second ordre relative aux champs de vitesse et température. L'analyse permet de considérer une température pariétale variable suivant une fonction de puissance de la distance quand il n'y a pas de dissipation, cependant quand il y a dissipation, la variation de température à la paroi est déterminée par le paramètre de gradient de pression si on recherche encore des solutions de similitude. Les équations différentielles ordinaires obtenues à partir de l'analyse de similarité sont résolues numériquement. On a présenté graphiquement les profils de température calculés au second ordre en fonction du paramètre de gradient de pression, du nombre de Prandtl, du nombre d'Eckert, du paramètre de distribution de la température pariétale et de la courbure de la surface. On voit que l'effet de second ordre est considérable pour des conditions proches de la séparation et qu'il n'est pas nécessairement négligeable comparé à celui du premier ordre. La dissipation peut considérablement affecter le transfert thermique pour les fluides à grands nombres de Prandtl, le nombre de Nusselt prenant des valeurs positives ou négatives quand le nombre d'Eckert varie de 0 à 1. Pour un nombre quelconque de Prandtl donné, lorsque la courbure passe du concave au convexe, le nombre de Nusselt décroît si le nombre d'Eckert est petit, tandis qu'il croît si le nombre d'Eckert est proche de l'unité.

EINFLUSS DER OBERFLÄCHENLÄNGSKRÜMMUNG AUF DEN WÄRMEÜBERGANG MIT DISSIPATION

Zusammenfassung—Es wird der Wärmeübergang mit Dissipation an einer Oberfläche, die in Strömungsrichtung gekrümmt ist, für erzwungene laminare Strömung mit der Methode der angepassten asymptotischen Entwicklungen untersucht. Mit der Lösung der klassischen Falkner-Skan-Eckenströmungen als erster Nährung für die Impulsgleichung wurde eine Lösung erster Näherung für die Energiegleichung mit Berücksichtigung der Dissipation ermittelt. Durch Ausdehnung der Betrachtungsweise wurde dann die Störung zweiter Ordnung für das Geschwindigkeits- und Temperaturfeld ermittelt. Das Verfahren gestattet, die Wandtemperatur als Potenzfunktion des Abstandes zu variieren, wenn man die Dissipation vernachlässigt. Wenn man jedoch die Dissipation berücksichtigt, wird die Änderung der Wandtemperatur bestimmt durch den Parameter des Druckgradienten, wenn man noch ähnliche Lösungen fordert. Die gewöhnlichen Differentialgleichungen aus der Ähnlichkeitsbetrachtung werden numerisch gelöst. Die in zweiter Näherung berechneten Temperaturprofile wurden graphisch aufgetragen als Funktionen des Parameters der Druckgradienten, der Prandtl-Zahl, der Eckert-Zahl, der Wandtemperaturverteilung und der Oberflächenkrümmung. Es wird gezeigt, dass der Effekt zweiter Ordnung beachtlich ist für die Bedingungen der Ablösung und nicht ohne weiteres vernachlässigt werden kann im Vergleich mit den Effekten erster Ordnung. Die Dissipation kann den Wärmeübergang in Medien mit hoher Prandtl-Zahl beachtlich beeinflussen, wobei die Nusselt-Zahl von positiven auf negative Werte überwechselt, wenn die Eckert-Zahl von Null auf Eins anwächst. Weiterhin nimmt bei gegebener Prandtl-Zahl die Nusselt-Zahl beim Übergang von einer konkaven zu einer konvexen Krümmung der Oberfläche ab, wenn die Eckert-Zahl klein ist, wogegen sie zunimmt, wenn die Eckert-Zahl in die Nähe von Eins kommt.

ВЛИЯНИЕ ПРОДОЛЬНОЙ КРИВИЗНЫ ПОВЕРХНОСТИ НА ТЕПЛООБМЕН С ДИССИПАЦИЕЙ

Аннотация—Анализировался теплообмен на поверхности с продольной кривизной с учетом диссипации для случая вынужденного течения методом сращиваемых асимптотических разложений. Используя классические клиновые течения Фолкнера-Скэна в качестве решений первого приближения уравнения количества движения, получено решение первого приближения уравнения энергии с учетом диссипации. Затем, продолжая этот анализ, получены решения второго приближения скоростных и температурных полей. Согласно этому анализу температура стенки может изменяться как степенная функция расстояния при отсутствии диссипации. Однако с учетом диссипации изменение температуры стенки определяется параметром градиента давления, если необходимы автомодельные решения. Численно решены обыкновенные дифференциальные уравнения, полученные из анализа подобия. Рассчитанные температурные профили второго приближения представлены графически как функции параметра градиента давления, числа Прандтля, числа Эккерта, параметра распределения температуры стенки и кривизны поверхности. Видно, что влияние второго приближения существенно для условий, близких к отрыву, и необязательно мало по сравнению с влиянием первого приближения. Диссипация может оказывать значительное влияние на теплообмен в жидкостях с большими числами Прандтля, при этом число Нуссельта изменяется от положительного значения до отрицательного по мере того, как число Эккерта изменяется от нуля до единицы. Далее, при любом заданном числе Прандтля с изменением кривизны от вогнутой до выпуклой число Нуссельта уменьшается при малом числе Эккерта и увеличивается, если число Эккерта близко к единице.